Differentiation: Product Rule

The **Product Rule** is used when we want to differentiate a function that may be regarded as a product of one or more simpler functions.
If our function $f(x) = g(x)h(x)$, where g and h are simpler functions, then The Product Rule may be stated as

$$\boxed{f'(x) = g'(x)h(x) + g(x)h'(x) \qquad \text{or} \qquad \frac{df}{dx}(x) = \frac{dg}{dx}(x)h(x) + g(x)\frac{dh}{dx}(x).}$$

In words, this says that to differentiate a product, we add the derivative of the first times the second to the first times the derivative of the second.

Example 1: Find the derivative of $f(x) = (x^3 + 2x^2 + x - 5)(-2x^3 + 4x^2 - 5x + 2)$.

Solution 1: In this case we could proceed by multiplying out the product and then differentiating the result. However this has a couple of drawbacks in that it would take quite a lot of work and also the chance of making a mistake is high. A much better method is to use the Product Rule. So let our functions g and h be

$$g(x) = x^3 + 2x^2 + x - 5 \qquad \text{and} \qquad h(x) = -2x^3 + 4x^2 - 5x + 2.$$

Then

$$g'(x) = 3x^2 + 4x + 1 \qquad \text{and} \qquad h'(x) = -6x^2 + 8x - 5.$$

Next, using the Product Rule, we see that the derivative of f is

$$f'(x) = g'(x)h(x) + g(x)h'(x) = (3x^2 + 4x + 1)(-2x^3 + 4x^2 - 5x + 2) + (x^3 + 2x^2 + x - 5)(-6x^2 + 8x - 5).$$

As an exercise, try multiplying out this expression for the derivative of f, then multiply out the expression for f, differentiate it and check that you get the same result. If you do this you will see that the Product Rule makes the differentiation **MUCH** easier.

Example 2: Find the derivative of $f(x) = (x^2 - 1)(2\cos 3x)$.

Solution 2: In this case we don't have any choice, we have to use the Product Rule; even if we multiply out the brackets, we will still end up with a product $2x^2 \cos 3x$.
So let our functions g and h be

$$g(x) = x^2 - 1 \qquad \text{and} \qquad h(x) = 2\cos 3x.$$

Then

$$g'(x) = 2x \qquad \text{and} \qquad h'(x) = -6\sin 3x.$$

Next, using the Product Rule, we see that the derivative of f is

$$f'(x) = g'(x)h(x) + g(x)h'(x) = 2x(2\cos 3x) + (x^2 - 1)(-6\sin 3x) = 4x\cos 3x - 6(x^2 - 1)\sin 3x.$$

Example 3: Find the derivative of $f(x) = 5e^{-3x}(\cos 3x - 5\sin 2x)$.

Solution 3: Let our functions g and h be

$$g(x) = 5e^{-3x} \qquad \text{and} \qquad h(x) = \cos 3x - 5\sin 2x.$$

Then

$$g'(x) = -15e^{-3x} \qquad \text{and} \qquad h'(x) = -3\sin 3x - 10\cos 2x.$$

Next, using the Product Rule, we see that the derivative of f is

$$\begin{aligned} f'(x) = g'(x)h(x) + g(x)h'(x) &= -15e^{-3x}(\cos 3x - 5\sin 2x) + 5e^{-3x}(-3\sin 3x - 10\cos 2x) \\ &= 5e^{-3x}(-3\cos 3x + 15\sin 2x - 3\sin 3x - 10\cos 2x). \end{aligned}$$

Example 4: Find the derivative of $f(x) = e^x \sin x + \ln x(\cos x)$.

Solution 4: Here we have a sum of products, so we have to use the Product Rule twice, once for each product, and then add the results.

First we will differentiate $u(x) = e^x \sin x$. Let our functions g and h be

$$g(x) = e^x \qquad \text{and} \qquad h(x) = \sin x.$$

Then

$$g'(x) = e^x \qquad \text{and} \qquad h'(x) = \cos x.$$

Next, using the Product Rule, we see that the derivative of u is

$$u'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x).$$

We now have to differentiate $v(x) = \ln x(\cos x)$. Let our functions g and h be

$$g(x) = \ln x \qquad \text{and} \qquad h(x) = \cos x.$$

Then

$$g'(x) = \frac{1}{x} \qquad \text{and} \qquad h'(x) = -\sin x.$$

Next, using the Product Rule, we see that the derivative of v is

$$v'(x) = \frac{1}{x}\cos x + \ln x(-\sin(x)) = \frac{\cos x}{x} - \ln x(\sin x).$$

Hence the derivative of f is

$$f'(x) = u'(x) + v'(x) = e^x(\sin x + \cos x) + \frac{\cos x}{x} - \ln x(\sin x).$$

Example 5: Find the derivative of $f(x) = x^2 e^x \sin x$.

Solution 5: Here we have a product of three functions, so again we have to use the Product Rule twice. In general it doesn't matter if we express f as $f(x) = x^2(e^x \sin x)$ or $f(x) = (x^2 e^x)\sin x$, but in this case we have already calculated the derivative of $e^x \sin x$ in Example 4, so we will use the Product Rule with

$$g(x) = x^2 \qquad \text{and} \qquad h(x) = e^x \sin x.$$

Then, using Example 4,

$$g'(x) = 2x \qquad \text{and} \qquad h'(x) = e^x(\sin x + \cos x).$$

Next, using the Product Rule, we see that the derivative of f is

$$\begin{aligned} f'(x) = g'(x)h(x) + g(x)h'(x) &= 2xe^x \sin x + x^2 e^x(\sin x + \cos x) \\ &= xe^x(2\sin x + x\sin x + x\cos x) \\ &= xe^x((2 + x)\sin x + x\cos x). \end{aligned}$$